Inner Product and Orthogonality Linear Algebra X. Du

- Generalization of the dot product
- **Inner product** is an operator to two vectors in a vector space that yields a scalar
- Axioms
    - $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
    - $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
    - $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$
    - $\langle \vec{v}, \vec{v} \rangle \geq 0$
    - $\langle \vec{v}, \vec{v} \rangle = 0$ iff $\vec{v} = \vec{0}$
- **Norm**: $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$
- **Distance**: $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$
- For Euclidean vectors, the inner product is usually the dot product
- For single variable functions, the inner product is usually $\langle f, g \rangle = \int_a^b f(x)g(x)dx$
- **Cauchy-Schwarz Inequality**: $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\|\|\vec{v}\|$
- **Orthogonality**: $\vec{u}, \vec{v}$ are orthogonal iff $\langle \vec{u}, \vec{v} \rangle = 0$
    - Orthogonal non-zero vectors are always linearly independent
- **Orthogonal basis**: a basis of V of only orthogonal vectors
    - Let $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ be an orthogonal basis for V. Then
      $\forall \vec{v} \in V, \vec{v} = \frac{\langle \vec{v}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\langle \vec{v}, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 + ... + \frac{\langle \vec{v}, \vec{v}_n \rangle}{\|\vec{v}_n\|^2}\vec{v}_n$
    - Note that this is just a bunch of vector projections
- **Orthonormal basis**: a basis of V of only orthogonal vectors with a norm of 1.
    - Let $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ be an orthonormal basis for V. Then
      $\forall \vec{v} \in V, \vec{v} = \langle \vec{v}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{v}, \vec{v}_2 \rangle \vec{v}_2 + ... + \langle \vec{v}, \vec{v}_n \rangle \vec{v}_n$
- **Orthogonal matrix**
    - Denoted Q.
    - A square matrix is an **orthogonal matrix** if its column vectors form an orthonormal set.
    - Row vectors also form an orthonormal set
    - A square matrix is orthogonal iff $Q^T = Q^{-1}$
        - Inverse of Q is also orthogonal
    - $\det(Q) = \pm 1$
    - Products of orthogonal matrices are orthogonal
    - $\|Q\vec{x}\| = \|\vec{x}\|$
    - $Q\vec{x} \cdot Q\vec{y} = \vec{x} \cdot \vec{y}$